▶ **Example 4.** Write $1/[2(\cos 20° + i\sin 20°]$ in $x+iy$ form. Since $20° = \pi/9$ radians,

$$\frac{1}{2(\cos 20° + i\sin 20°)} = \frac{1}{2(\cos\pi/9 + i\sin\pi/9)} = \frac{1}{2\,e^{i\pi/9}} = 0.5\,e^{-i\pi/9}$$
$$= 0.5(\cos\pi/9 - i\sin\pi/9) = 0.47 - 0.17i,$$

by calculator in radian mode. We obtain the same result leaving the angle in degrees and using a calculator in degree mode: $0.5(\cos 20° - i\sin 20°) = 0.47 - 0.17i$.

▶ PROBLEMS, SECTION 5

First simplify each of the following numbers to the $x+iy$ form or to the $re^{i\theta}$ form. Then plot the number in the complex plane.

1. $\dfrac{1}{1+i}$ **2.** $\dfrac{1}{i-1}$ **3.** i^4

4. i^2+2i+1 **5.** $\left(i+\sqrt{3}\right)^2$ **6.** $\left(\dfrac{1+i}{1-i}\right)^2$

7. $\dfrac{3+i}{2+i}$ **8.** $1.6-2.7i$

9. $25\,e^{2i}$ *Careful!* The angle is 2 radians.

10. $\dfrac{3i-7}{i+4}$ *Careful! Not* $3-7i$

11. $17-12i$ **12.** $3(\cos 28° + i\sin 28°)$

13. $5\left(\cos\dfrac{2\pi}{5} + i\sin\dfrac{2\pi}{5}\right)$ **14.** $2.8e^{-i(1.1)}$

15. $\dfrac{5-2i}{5+2i}$ **16.** $\dfrac{1}{0.5(\cos 40° + i\sin 40°)}$

17. $(1.7-3.2i)^2$ **18.** $(0.64+0.77i)^4$

Find each of the following in rectangular $(a+bi)$ form if $z = 2-3i$; if $z = x+iy$.

19. z^{-1} **20.** $\dfrac{1}{z^2}$ **21.** $\dfrac{1}{z+1}$

22. $\dfrac{1}{z-i}$ **23.** $\dfrac{1+z}{1-z}$ **24.** $z/\bar{z}$

B. Complex Conjugate of a Complex Expression

It is easy to see that the conjugate of the sum of two complex numbers is the sum of the conjugates of the numbers. If

$$z_1 = x_1 + iy_1 \qquad \text{and} \qquad z_2 = x_2 + iy_2,$$

then

$$\bar{z}_1 + \bar{z}_2 = x_1 - iy_1 + x_2 - iy_2 = x_1 + x_2 - i(y_1 + y_2).$$

The conjugate of $(z_1 + z_2)$ is

$$\overline{(x_1+x_2) + i(y_1+y_2)} = (x_1+x_2) - i(y_1+y_2).$$

Similarly, you can show that the conjugate of the difference (or product or quotient) of two complex numbers is equal to the difference (or product or quotient) of the conjugates of the numbers (Problem 25). In other words, you can get the conjugate of an expression containing i's by just changing the signs of all the i terms. We must watch out for hidden i's, however.

▶ **Example.** If

$$z = \frac{2-3i}{i+4}, \qquad \text{then} \qquad \bar{z} = \frac{2+3i}{-i+4}.$$

But if $z = f + ig$, where f and g are themselves complex, then the complex conjugate of z is $\bar{z} = \bar{f} - i\bar{g}$ (*not* $f - ig$).

▶ PROBLEMS, SECTION 5

25. Prove that the conjugate of the quotient of two complex numbers is the quotient of the conjugates. Also prove the corresponding statements for difference and product. *Hint:* It is easier to prove the statements about product and quotient using the polar coordinate $re^{i\theta}$ form; for the difference, it is easier to use the rectangular form $x + iy$.

C. Finding the Absolute Value of z

Recall that the definition of $|z|$ is $|z| = r = \sqrt{x^2+y^2}$ (positive square root!). Since $z\bar{z} = (x+iy)(x-iy) = x^2+y^2$, or, in polar coordinates, $z\bar{z} = (re^{i\theta})(re^{-i\theta}) = r^2$, we see that $|z|^2 = z\bar{z}$, or $|z| = \sqrt{z\bar{z}}$. Note that $z\bar{z}$ is always real and ≥ 0, since x, y, and r are real. We have

$$|z| = r = \sqrt{x^2+y^2} = \sqrt{z\bar{z}}. \tag{5.1}$$

By Problem 25 and (5.1), the absolute value of a quotient of two complex numbers is the quotient of the absolute values (and a similar statement for product).

▶ **Example.**

$$\left|\frac{\sqrt{5}+3i}{1-i}\right| = \frac{|\sqrt{5}+3i|}{|1-i|} = \frac{\sqrt{14}}{\sqrt{2}} = \sqrt{7}.$$

▶ PROBLEMS, SECTION 5

Find the absolute value of each of the following using the discussion above. Try to do simple problems like these in your head—it saves time.

26. $\dfrac{2i-1}{i-2}$

27. $\dfrac{2+3i}{1-i}$

28. $\dfrac{z}{\bar{z}}$

29. $(1+2i)^3$

30. $\dfrac{3i}{i-\sqrt{3}}$

31. $\dfrac{5-2i}{5+2i}$

32. $(2-3i)^4$

33. $\dfrac{25}{3+4i}$

34. $\left(\dfrac{1+i}{1-i}\right)^5$

D. Complex Equations

In working with equations involving complex quantities, we must always remember that a complex number is actually a pair of real numbers. Two complex numbers are equal if and only if their real parts are equal and their imaginary parts are equal. For example, $x + iy = 2 + 3i$ means $x = 2$ and $y = 3$. In other words, any equation involving complex numbers is really two equations involving real numbers.

▶ **Example.** Find x and y if

$$(x + iy)^2 = 2i. \tag{5.2}$$

Since $(x + iy)^2 = x^2 + 2ixy - y^2$, (5.2) is equivalent to the two real equations

$$x^2 - y^2 = 0,$$
$$2xy = 2.$$

From the first equation $y^2 = x^2$, we find $y = x$ or $y = -x$. Substituting these into the second equation gives

$$2x^2 = 2 \qquad \text{or} \qquad -2x^2 = 2.$$

Since x is real, x^2 cannot be negative. Thus we find only

$$x^2 = 1 \qquad \text{and} \qquad y = x,$$

that is,

$$x = y = 1 \qquad \text{and} \qquad x = y = -1.$$

▶ PROBLEMS, SECTION 5

Solve for all possible values of the real numbers x and y in the following equations.

35. $x + iy = 3i - 4$

36. $2ix + 3 = y - i$

37. $x + iy = 0$

38. $x + iy = 2i - 7$

39. $x + iy = y + ix$

40. $x + iy = 3i - ix$

41. $(2x - 3y - 5) + i(x + 2y + 1) = 0$

42. $(x + 2y + 3) + i(3x - y - 1) = 0$

43. $(x + iy)^2 = 2ix$

44. $x + iy = (1 - i)^2$

45. $(x + iy)^2 = (x - iy)^2$

46. $\dfrac{x + iy}{x - iy} = -i$

47. $(x + iy)^3 = -1$

48. $\dfrac{x + iy + 2 + 3i}{2x + 2iy - 3} = i + 2$

49. $|1 - (x + iy)| = x + iy$

50. $|x + iy| = y - ix$

E. Graphs

Using the graphical representation of the complex number z as the point (x, y) in a plane, we can give geometrical meaning to equations and inequalities involving z.

▶ **Example 1.** What is the curve made up of the points in the (x, y) plane satisfying the equation $|z| = 3$?

Since

$$|z| = \sqrt{x^2 + y^2},$$

the given equation is

$$\sqrt{x^2 + y^2} = 3 \quad \text{or} \quad x^2 + y^2 = 9.$$

Thus $|z| = 3$ is the equation of a circle of radius 3 with center at the origin. Such an equation might describe, for example, the path of an electron or of a satellite. (See Section F below.)

▶ **Example 2.**

(a) $|z - 1| = 2$. This is the circle $(x - 1)^2 + y^2 = 4$.

(b) $|z - 1| \le 2$. This is the disk whose boundary is the circle in (a).

Note that we use "circle" to mean a curve and "disk" to mean an area. The interior of the disk is given by $|z - 1| < 2$.

▶ **Example 3.** (Angle of z) $= \pi/4$. This is the half-line $y = x$ with $x > 0$; this might be the path of a light ray starting at the origin.

▶ **Example 4.** $\operatorname{Re} z > \frac{1}{2}$. This is the half-plane $x > \frac{1}{2}$.

▶ PROBLEMS, SECTION 5

Describe geometrically the set of points in the complex plane satisfying the following equations.

51. $|z| = 2$

52. $\operatorname{Re} z = 0$

53. $|z - 1| = 1$

54. $|z - 1| < 1$

55. $z - \bar{z} = 5i$

56. angle of $z = \dfrac{\pi}{2}$

57. $\operatorname{Re}(z^2) = 4$

58. $\operatorname{Re} z > 2$

59. $|z + 3i| = 4$

60. $|z - 1 + i| = 2$

61. $\operatorname{Im} z < 0$

62. $|z + 1| + |z - 1| = 8$

63. $z^2 = \bar{z}^2$

64. $z^2 = -\bar{z}^2$

65. Show that $|z_1 - z_2|$ is the distance between the points z_1 and z_2 in the complex plane. Use this result to identify the graphs in Problems 53, 54, 59, and 60 without computation.

F. Physical Applications

Problems in physics as well as geometry may often be simplified by using one complex equation instead of two real equations. See the following example and also Section 16.